An introduction to the Riemann-Hilbert correspondence

⚠ Various versions exist.

Our context: $(X, \mathcal{O}_X)$ a <u>geometric space</u>.

locally ringed space $(X, \mathcal{O}_X)$

[i.e. $\forall x \in X$, $\mathcal{O}_{X,x}$ is local

and $(f, \varphi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$

satisfies $\varphi_x(m_{Y,f(x)}) \subset m_{X,x}$]

<u>Example to keep in mind</u> (at all cost!)

$(X, \mathcal{O}_X)$: complex analytic manifold

$\rightsquigarrow$ more rigid than a $\mathcal{C}^\infty$ manifold, but the topology is easier to handle than that of a complex algebraic variety

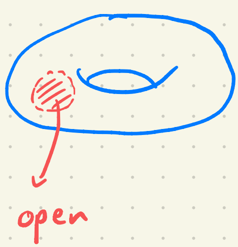
SLOGAN:

"Always start with the complex one-dimensional case"

$\rightsquigarrow$ X could be a non-singular complex projective curve, e.g. a smooth plane cubic

$$\Big\{ [x:y:z] \in \mathbb{CP}^2 \;\Big|\; \underbrace{y^2 z - x(x+1)(x-1) = 0}_{\text{actually defined over } \mathbb{R}} \Big\}$$

but here we think of it analytically and picture it like this:

$X \simeq$ 

$\mathcal{O}_X$:= sheaf of holomorphic functions on X (makes sense)

Last time, we saw a correspondence:

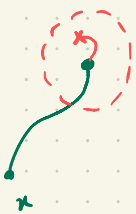
$$\begin{array}{ccc} \mathrm{Et}/X & \longleftrightarrow & \mathrm{Sh}_X \\ \cup & & \cup \\ \mathrm{Cov}/X & \longleftrightarrow & \mathcal{L}oc_X \end{array}$$

⇝ Two equivalent ways to think about locally constant sheaves:

- $\mathcal{F} \in \mathrm{Sh}_X$ s.t. $\mathcal{F}|_U \simeq \underline{F}_U$ for some set F

- sheaf of continuous sections of a $\underbrace{\text{topological covering space } Y \to X}$

if X is a connected manifold, these are "classified" by $\pi_1(X, x_0)$



$$\begin{array}{ccc} \mathrm{Cov}/X & \longleftrightarrow & \pi_1(X,x_0)\text{-Sets} \\ (Y \xrightarrow{p} X) & \longmapsto & p^{-1}(\{x_0\}) \\ (\tilde{X}_{x_0} \times F)/\pi_1 X & \longleftarrow & F \text{ (discrete top.)} \end{array}$$

Obs: if Y is connected, we can also set $Y \longmapsto \pi_1 Y$ as a functor (to subgps of $\pi_1(X, x_0)$)

If one restricts to covering spaces with a given fibre F, one gets a correspondence:

$$\left\{\begin{array}{c}\text{locally constant} \\ \text{sheaves on } X \\ \text{with stalk } F\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{covering spaces} \\ \text{of } X \text{ with} \\ \text{fibre } F\end{array}\right\}$$

$$\longleftrightarrow \left\{\begin{array}{c}\text{representations} \\ \rho : \pi_1 X \to \mathrm{Aut}(F)\end{array}\right\}$$

Example $F = \mathbb{C}^n$

$$\left\{\begin{array}{c}\text{locally constant} \\ \text{sheaves of} \\ n\text{-dimensional} \\ \text{complex vector} \\ \text{spaces on } X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{linear representations} \\ \rho : \pi_1 X \to GL(n;\mathbb{C})\end{array}\right\}$$

For instance, when $\dim_{\mathbb{C}} X = 1$, one could think of the sheaf of solutions to a linear ODE with holomorphic coefficients $Y'(z) = A(z)\,Y(z)$.

The previous example tells us that there can be a third point of view:

$\underbrace{\text{vector bundles}}$ with connection.

$p: E \to X$ such that $\forall x \in X$

(i) $p^{-1}(\{x\})$ is a vector space

(ii) $\exists$ open neighbourhood $U_x \ni x$ and a local trivialisation

$$\begin{tikzcd} p^{-1}(U_x) \arrow[rr] \arrow[dr, "p"'] & & U_x \times p^{-1}(\{x\}) \arrow[dl, "pr_1"] \\ & U_x & \end{tikzcd}$$

that is linear in the fibres.

$\rightsquigarrow$ This definition encapsulates the intuition that a vector bundle on X is a locally trivial family of vector spaces parameterised by X.

Equivalently, $p: E \to X$ is a vector bundle if there exists an open covering $(U_i)_{i \in I}$ of X, and local trivialisations

$$\begin{tikzcd} \Phi_i : p^{-1}(U_i) \arrow[rr, "\simeq"] \arrow[dr, "p"'] & & U_i \times V_i \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

where V_i is a vector space, such that, for all (i,j), the isomorphism

$$\Phi_j \circ \Phi_i^{-1} : (U_i \cap U_j) \times V_i \xrightarrow{\simeq} (U_i \cap U_j) \times V_j$$

induces, for all $x \in U_i \cap U_j$, a <u>linear</u> isomorphism

$$g_{ji}(x) : V_i \longrightarrow V_j .$$

$\rightsquigarrow$ this is usually how one proves that a given $p: E \to X$ defines a vector bundle (by <u>constructing</u> it by <u>gluing</u>).

Exercise Let $(X, \mathcal{O}_X)$ be an n-dimensional complex analytic manifold and let

$$TX := \bigsqcup_{x \in X} T_x X$$

where

$$T_x X := \mathrm{Hom}_{k(x)\text{-mod}} \Big(\underbrace{m_{X,x} / m_{X,x}^2 \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x} / m_{X,x}}_{\text{or directly } m_{X,x}/m_{X,x}^2} \, ; \, k(x) \Big)$$

for $m_{X,x} \subset \mathcal{O}_{X,x}$ the maximal ideal and $k(x) := \mathcal{O}_{X,x} / m_{X,x}$ the residue field.

with k-module structure induced by $\mathcal{O}_{X,x} \hookrightarrow k \to 0$

Show that TX is a vector bundle on X, with fibre $\simeq k(x)^n$.

Indication Use an atlas $(U_i, \varphi_i)_{i \in I}$ of X and use the U_i as trivialising open sets for TX.

Claim 1 If $E \to X$ is a vector bundle of rank n, its sheaf of sections is a locally free $\mathcal{O}_X$-module of rank n: over a trivialising open set U, the <u>sheaf</u> $\Gamma_{E|_U}$ is isomorphic to $\mathcal{O}_U^n$:

$$\begin{tikzcd} E|_U \arrow[rr, "\Phi"] \arrow[dr, "p"'] & & U \times k^n \arrow[dl, "pr_1"] \\ & U \arrow[ul, bend left, "s"] & \end{tikzcd}$$

section: $s \mid p \circ s = id_U$

$$s \longleftrightarrow \Phi \circ s = (id_U, f)$$

$$f : U \to k^n$$

$k^n = k e_1 \oplus \cdots \oplus k e_n$ (canonical basis)

Define $\hat{e_i} : U \to k^n$
$x \mapsto e_i$ (constant map)

and $s_i := \Phi^{-1} \circ (id_U, e_i)$

$(s_1, \ldots, s_n)$: family of n sections of p such that, $\forall x \in U$, $(s_1(x), \ldots, s_n(x))$ is a basis of $E_x := p^{-1}(\{x\})$.

Then:

$$\forall\, U' \subset U, \quad \Gamma_E(U') \simeq \underbrace{\mathcal{O}_X(U')s_1 \oplus \dots \oplus \mathcal{O}_X(U')s_n}_{\text{free } \mathcal{O}_X(U')\text{-module of rank } n}$$

trivialising open set for E

Claim 2 Conversely, to a locally free $\mathcal{O}_X$-module $\mathcal{M}$, there is associated a vector bundle E, such that Γ_E is <u>locally</u> isomorphic to $\mathcal{M}$ ($\Gamma_E|_{U_i} \simeq \mathcal{M}|_{U_i}$ for some open covering $(U_i)_{i\in I}$ of X).

Observation By assumption on $\mathcal{M}$ $\exists$ an isomorphism of sheaves on U_i

$$\Phi_i : \mathcal{M}|_{U_i} \xrightarrow{\simeq} \mathcal{O}^n_{U_i}.$$

Then $\Phi_j \circ \Phi_i^{-1} : \mathcal{O}^n_{U_i\cap U_j} \longrightarrow \mathcal{O}^n_{U_i \cap U_j}$

is an isomorphism of free $\mathcal{O}_{U_i\cap U_j}$-modules.

we often write

$$\underbrace{\Phi_j \circ \Phi_i^{-1} \in GL(n, \mathcal{O}_{U_i \cap U_j})}$$

a sheaf

to abbreviate this. In particular, over $U_i \cap U_j$, we have an element

$$g_{ji} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

with $a_{ij} \in \mathcal{O}_X(U_i \cap U_j)$

holomorphic function
regular

so g_{ji} is a matrix-valued function such that $\forall x \in U_i \cap U_j$, $g_{ji}(x) \in GL(n; k)$.

In fact:

$$g_{ji} \in GL(n; \underbrace{\mathcal{O}_X(U_i \cap U_j)}_{\text{a ring}}).$$

Point We can use the functions

$$g_{ji} : U_i \cap U_j \longrightarrow GL(n;\mathbb{C})$$

to construct a vector bundle

$$E = \left(\bigsqcup_{i \in I} U_i \times \mathbb{C}^n \right) \Big/ \begin{array}{l} (x_i, v_i) \sim (x_j, v_j) \\ \text{if } x_i = x_j \\ \text{and } v_j = g_{ji}(x_i) v_i \end{array}$$

This vector bundle comes equipped with local trivialisations $E|_{U_i} \simeq U_i \times \mathbb{C}^n$,

so $\Gamma_{E|_{U_i}} \simeq \mathcal{O}^n_{U_i} \simeq \mathcal{M}|_{U_i}$. In particular, $(\Gamma_E)_x \simeq \mathcal{M}_x$ as an $\mathcal{O}_{X,x}$-module. And

the <u>fibre</u> E_x is obtained as the $\mathbb{C}(x)$-vector space

$$\mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \underbrace{\mathcal{O}_{X,x} / m_{X,x}}_{\mathbb{C}(x)}$$

exercise

via the evaluation map $ev_x : (\Gamma_E)_x \to E_x$.

Note that $\mathcal{M}_x \simeq \mathcal{O}_{X,x}^n$ and that k is an $\mathcal{O}_{X,x}$-module via the evaluation map

$$ev_x : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = k,$$

so $\mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} k \simeq \mathcal{O}_{X,x}^n \otimes_{\mathcal{O}_{X,x}} k \simeq k^n$.

We have thus defined two (functorial) correspondences

$$\left\{ \begin{array}{c} \text{vector bundles} \\ \text{on } X \end{array} \right\} \underset{\longleftarrow}{\overset{\Gamma}{\longrightarrow}} \left\{ \begin{array}{c} \text{locally free} \\ \mathcal{O}_X\text{-modules} \end{array} \right\}$$

$$E \longmapsto \Gamma_E$$

$$\bigsqcup_{x \in X} \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \left(\mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \right) \longleftarrow\!\!\shortmid \mathcal{M}$$

Theorem These two functors are quasi-inverse to each other. In particular, the above correspondence is an equivalence of categories.

Observation The constructions above make sense (and the theorem still holds) if $(X, \mathcal{O}_X)$ is a smooth algebraic variety.

From now, we identify vector bundles on X with locally free $\mathcal{O}_X$-modules via the above correspondence.
I will use the notation $\mathcal{E}$ to mean Γ_E (the sheaf of sections of a vector bundle).

Example

$$\Omega^1_X := \left\{ \begin{array}{l} \text{sheaf of (regular) sections of} \\ \\ T^*X := \bigsqcup_{x \in X} m_{X,x}/m_{X,x}^2 \end{array} \right.$$

these sections are called (regular) 1-forms

For instance, if f is a regular function, its differential df is a 1-form.

Claim Ω^1_X is a locally free sheaf.

Definition A linear connection (or covariant derivative) on a vector bundle $\mathcal{E}$ over $(X, \mathcal{O}_X)$ is a morphism of sheaves of Abelian groups

$$\mathcal{E} \xrightarrow{\nabla} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$$

such that "$\forall f \in \mathcal{O}_X$", "$\forall s \in \mathcal{E}$",

Leibniz identity

$$\nabla(\underbrace{fs}_{\mathcal{E} \text{ is an } \mathcal{O}_X\text{-module}}) = df \otimes s + \underbrace{f \nabla s}_{\substack{\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E} \text{ is an } \mathcal{O}_X\text{-module} \\ \text{(tensor product of two } \mathcal{O}_X\text{-modules)}}}$$

∇ is <u>not</u> a morphism of $\mathcal{O}_X$-modules! The condition for that would be $\theta : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ such that $\theta(fs) = f\theta(s)$. Such a θ is called a Higgs field.

Example Let V be a locally constant sheaf of vector spaces on X.

third point on locally constant sheaves of vector spaces

Based on all that we have done, we can view V as the sheaf of continuous sections of a locally free $\mathcal{C}st_X$-module, where $\mathcal{C}st_X$ is the sheaf of k-valued locally constant functions on X.

Note that $\mathcal{O}_X$ is a $\mathcal{C}st_X$-module ("locally constant functions are regular").

Then $\mathcal{E} := \mathcal{O}_X \otimes_{\mathcal{C}st_X} V$ is a locally free $\mathcal{O}_X$-module. Moreover, we can equip $\mathcal{E}$ with the linear connection

$$\nabla := d \otimes id_V$$

where $d : \begin{array}{l} \mathcal{O}_X \to \Omega^1_X \\ f \mapsto df \end{array}$.

$$\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E} \simeq \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \otimes_{\mathcal{C}st_X} V \simeq \Omega^1_X \otimes_{\mathcal{C}st_X} V$$

Leibniz: $\nabla(f(g\otimes s)) = \nabla((fg)\otimes s) = d(fg)\otimes s$
$= (df\, g + f\, dg)\otimes s = df \otimes (g\otimes s) + f\,\nabla(g\otimes s)$.

The previous example shows that there is a functor

$$\left\{\begin{array}{c}\text{locally free}\\ \mathcal{C}st_X\text{-modules}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{locally free}\\ \mathcal{O}_X\text{-modules}\\ \text{with connection}\end{array}\right\}$$

$$\mathcal{V} \longmapsto \left(\mathcal{O}_X \underset{\mathcal{C}st_X}{\otimes} \mathcal{V},\ d \otimes id_{\mathcal{V}}\right)$$

Is there one in the converse direction?

→ Recall the example from ODE theory, where the locally constant sheaf arises as the sheaf of solutions to an ODE.

$$Y'(z) = A(z)\,Y(z)$$

$$\Leftrightarrow \underbrace{dY - A(z)Y(z)\,dz}_{DY} = 0 \qquad dY = Y'(z)\,dz$$

Exercise Prove that "$D = d - A$" defines a linear connection on the trivial vector bundle $X \times \mathbb{C}^n$.

So now we set:

$$\left\{\begin{array}{l}\text{locally free}\\ \mathcal{E}st_X\text{-modules}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{locally free}\\ \mathcal{O}_X\text{-modules}\\ \text{with connection}\end{array}\right\}$$

$$\mathcal{V} \longmapsto \left(\mathcal{O}_X \otimes_{\mathcal{E}st_X} \mathcal{V},\ d \otimes id_{\mathcal{V}}\right)$$

$$\mathcal{E}^{\nabla} \longleftarrow\!\shortmid \underbrace{(\mathcal{E}, \nabla)}_{\text{too general}}$$

$$\downarrow$$

the sheaf defined by

$$\mathcal{E}^{\nabla}(U) = \{ s \in \mathcal{E}(U) \text{ such that } \underbrace{\nabla s = 0}_{\text{"horizontal sections"}} \}$$

i.e. $\mathcal{E}^{\nabla} = \ker \nabla$ (kernel of a morphism of sheaves of Abelian groups)

if f is loc. constant, $\nabla(fs) = \underbrace{df}_{=0} \otimes s + f \underbrace{\nabla s}_{=0}$

Issues:

- Is this indeed a locally free $\mathcal{E}st_X$-module?

  more complicated

- Is $rk_{\mathcal{E}st_X} \mathcal{V} = rk_{\mathcal{O}_X} \mathcal{E}$?

In other words, the issue is to find "enough linearly independent solutions to the equation $\nabla s = 0$".

Take-away This will hold under a certain "integrability condition" on ∇, which is always satisfied when $\dim_{\mathbb{C}} X = 1$ ...

Point

- The linear connection $\nabla = d \otimes \mathrm{id}_V$ from the example is always integrable.
- The integrability condition for the connection $\nabla = d - \omega$ (where $\omega = A(z)\, dz$) is the Maurer–Cartan equation

$$d\omega + \frac{1}{2}[\omega \wedge \omega] = 0.$$

"matrix of 1-forms"

This gives rise to the <u>Riemann-Hilbert</u> correspondence (in this setting).

$$\left\{\begin{array}{c}\text{locally constant}\\ \text{sheaves of vector}\\ \text{spaces on } X\end{array}\right\} \xleftrightarrow[\substack{\text{for } X\\ \text{a manifold}}]{} \left\{\begin{array}{c}\text{vector bundles}\\ \text{with \underline{integrable}}\\ \text{connection on } X\end{array}\right\}.$$

also called flat connection

for X a "nice" topological space $\Big\updownarrow$ RH

$$\left\{\begin{array}{c}\text{linear representations}\\ \rho : \pi_1 X \to GL(V)\end{array}\right\}$$